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# Global dimension of bi-amalgamated algebras along pure ideals[☆]

Khalid Louartiti [a,*], Mohammed Tamekkante [b]

[a] Department of Mathematics, Faculty of Science, BEN M'SIK, Box 7955, Sidi Othmam, University Hassan II, Casablanca, Morocco
[b] Department of Mathematics, Faculty of Science, Box 11201, Zitoune, University Moulay Ismail, Meknes, Morocco

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## Abstract

Let $f: A \to B$ and $g: A \to C$ be two ring homomorphisms and let $J$ and $J'$ be two ideals of $B$ and $C$, respectively, such that $f^{-1}(J) = g^{-1}(J')$. The bi-amalgamation of $A$ with $(B, C)$ along $(J, J')$ with respect to $(f, g)$ is the subring of $B \times C$ given by

$$A \bowtie^{f,g} (J, J\prime) = \{(f(a) + j, g(a) + j\prime) \mid a \in A, (j, j\prime) \in J \times J\prime\}$$

The aim of this paper is to characterize the global dimension of bi-amalgamated algebras over pure ideals.



*Keywords:* Bi-amalgamated algebras; Pure ideals; Global dimension of rings

## 1. Introduction

Throughout, all rings considered are commutative with unity and all modules are unital. For a ring $R$, gldim$(R)$ and wgldim$(R)$ will denote the global dimension of $R$ and the weak global dimension of $R$, respectively. For an $R$-module $M$, the projective dimension of $M$ and the flat dimension of $M$ are denoted by $\mathrm{pd}_R(M)$ and $\mathrm{fd}_R(M)$, respectively.

The following diagram of ring homomorphisms

$$\begin{tikzcd} R \arrow[r, "\iota_2"] \arrow[d, "\mu_2"'] & R_1 \arrow[d, "\mu_1"] \\ R_2 \arrow[r, "\iota_1"] & R' \end{tikzcd}$$

is called pullback (or fiber product) if the homomorphism $\iota_2 \times \mu_2 : R \to R_1 \times R_2$ induces an isomorphism of $R$ onto the subring of $R_1 \times R_2$ given by

$$\mu_1 \times \iota_1 := \{(r_1, r_2) \mid \mu_1(r_1) = \iota_1(r_2)\}$$

The homological properties of a fibre product $R$ have been studied previously. Milnor [1] has characterized projective modules over such a ring $R$ assuming that $\iota_1$ is surjective. Facchini and Vamos [2] have obtained analogues of Milnor's theorems for injective and flat

[☆] Supported by Laboratory of Mathematics, Computing and Application, University Mohammed V-Agdal, Rabat, Morocco.

* Corresponding author. Tel.: +212 606140602.
*E-mail addresses:* khalid.louartiti@usmba.ac.ma (K. Louartiti), tamekkante@yahoo.fr (M. Tamekkante).

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modules. In 1985, Wiseman [3] obtained the following upper bound on the global dimension of $R$:

$$\mathrm{gldim}(R) \leqslant \max\{\mathrm{gldim}(R_1), \mathrm{gldim}(R_2)\} + \max\{\mathrm{fd}_R(R_1), \mathrm{fd}_R(R_2)\}$$

He also pointed out the fact that it is impossible to estimate gldim($R$) with only gldim($R_1$) and gldim($R_2$) given, because there exists examples in which the pullback $R$ has infinite global dimension whilst those of the component rings $R_1$ and $R_2$ are finite.

In 1988, Kirkman and Kuzmanovich [4] showed that if $\iota_1$ is surjective then

$$\mathrm{gldim}(R) \leqslant \max\{\mathrm{gldim}(R_1) + \mathrm{fd}_R(R_1), \mathrm{gldim}(R_2) + \mathrm{fd}_R(R_2)\}$$

The aim of this paper is to give a preliminary study of the (weak) global dimension of a subclass of pullbacks rings called bi-amalgamated algebras introduced in [5].

Let $f: A \to B$ and $g: A \to C$ be two ring homomorphisms and let $J$ and $J'$ be two ideals of $B$ and $C$, respectively, such that $f^{-1}(J) = g^{-1}(J')$. The bi-amalgamation of $A$ with $(B, C)$ along $(J, J')$ with respect to $(f, g)$ is the subring of $B \times C$ given by

$$A \bowtie^{f,g}(J, J') = \{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J'\}$$

In [5], the authors studied ring-theoretic properties of bi-amalgamations, provided examples of bi-amalgamations, studied the transfer of some basic ring theoretic properties to bi-amalgamations and described the prime ideal structure of these constructions. They also showed how these constructions arise as pullbacks. Given $f: A \to B$ and $g: A \to C$ two ring homomorphisms and $J$ and $J'$ be two ideals of $B$ and $C$, respectively, such that $f^{-1}(J) = g^{-1}(J') := I$, the bi-amalgamation is determined by the following pullback:

$$\begin{tikzcd} A \bowtie^{f,g}(J, J') \arrow[r, two heads, "\mu_1"] \arrow[d, "\mu 2"'] & f(A) + J \arrow[d, "\alpha"] \\ g(A) + J' \arrow[r, "\beta"] & A/I \end{tikzcd}$$

where $\mu_1$ and $\mu_2$ are the surjection morphisms induced from the canonical surjections of $(f(A) + J) \times (g(A) + J')$ into $f(A) + J$ and $g(A) + J'$, respectively, and $\alpha(f(a) + j) = \overline{a}$ and $\beta(g(a) + j') = \overline{a}$, for each $a \in A$ and $j, j' \in J \times J'$. That is

$$A \bowtie^{f,g}(J, J') = \alpha \times_{\frac{A}{I}} \beta$$

The interest of these bi-amalgamations resides, partly, in their ability to cover several basic constructions in commutative algebra, including classical pullbacks (e.g., $D + M$, $A + XB[X]$, $A + XB[[X]]$, etc.), Nagata's idealizations (also called trivial ring extensions which have been widely studied in the literature), and Boisen–Sheldon's CPI-extensions [6].

Given a ring homomorphism $f: A \to B$ and an ideal $J$ of $B$, the bi-amalgamation $A \bowtie^{\iota,f}(f^{-1}(J), J)$ coincides with the amalgamated algebra introduced in 2009 by D'Anna, Finocchiaro, and Fontana [7,8] as the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A, \ j \in J\}$$

When $A = B$ and $f = \mathrm{id}$, the amalgamated $A \bowtie^{\mathrm{id}} I$ is called amalgamated duplication of a ring $A$ along the ideal $I$ and denoted $A \bowtie I$ (Introduced in 2007 by D'Anna and Fontana, [9]). This construction can be presented as a bi-amalgamated algebra as follows:

$$A \bowtie I = A \bowtie^{\mathrm{id},\mathrm{id}}(I, I)$$

This paper studies the (weak) global dimension of bi-amalgamated algebras over pure ideals and gives some results concerning the (weak) global dimension of the amalgamated duplication of a ring along an ideal.

Throughout, let $f: A \to B$ and $g: A \to C$ be two ring homomorphisms and let $J$, $J'$ two ideals of $B$ and $C$, respectively, such that $I := f^{-1}(J) = g^{-1}(J')$. Let

$$A \bowtie^{f,g}(J, J') = \{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J'\}$$

the bi-amalgamation of $A$ with $(B, C)$ along $(J, J')$ with respect to $(f, g)$.

## 2. Global dimension of bi-amalgamated algebras along pure ideals

Let $R$ be a ring and $M$ an $R$-module. A submodule $N$ of $M$ is called pure submodule of $M$ if for every $R$-module $L$ the sequence

$$0 \to N \otimes_R L \to M \otimes_R L \to M/N \otimes_R L \to 0$$

is exact. In particular, for $M = R$ and $I = N$ is an ideal of $R$, $I$ is called a pure ideal of $R$ [10]. Pure ideals of a rings $R$ are ideals $I$ such that $R/I$ is flat $R$-modules [10, Theorem 1.2.15].

The purpose of this section is to characterize the global dimension of bi-amalgamated algebras over pure ideals.

Consider the following pullback of rings:

$$\begin{array}{ccc} A \bowtie^{f,g} (J, J') & \xrightarrow{\mu_1} & f(A) + J \\ \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \alpha} \\ g(A) + J' & \xrightarrow{\beta} & A/I \end{array} \qquad (\square)$$

where $\mu_1$ and $\mu_2$ are the surjection morphisms induced from the canonical surjections of $(f(A)+J) \times (g(A)+J')$ into $f(A)+J$ and $g(A)+J'$, respectively, and $\alpha(f(a)+j)=\overline{a}$ and $\beta(g(a)+j\prime)=\overline{a}$, for each $a \in A$ and $j, j' \in J \times J'$.

Throughout, the rings $f(A)+J$ and $g(A)+J'$ are considered as $(A \bowtie f, g(J, J\prime))$-modules via $\mu_1$ and $\mu_2$, respectively.

**Lemma 2.1.** *We have the following isomorphisms of $(A \bowtie f, g(J, J\prime))$-modules:*

$$\frac{A \bowtie f, g(J, J\prime)}{0 \times J\prime} \cong f(A) + J \qquad \text{and}$$

$$\frac{A \bowtie f, g(J, J\prime)}{J \times 0} \cong g(A) + J\prime$$

**Proof.** It is easily seen that the surjective canonical map $\mu_1 : A \bowtie^{f,g}(J, J') \twoheadrightarrow f(A)+J$ is a homomorphism of $(A \bowtie f, g(J, J\prime))$-modules. If $f(a)+j=0$ for some $a \in A$ and $j \in J$, then $g(a)+j' \in J'$ for any $j' \in J'$. So the kernel of $\mu_1$ coincides with $0 \times J'$. Hence, the first isomorphism holds and the second one follows similarly. □

**Lemma 2.2** ([10],Theorem 1.2.15). *Let $A$ be a ring and let $I$ be an ideal of $A$. The following conditions are equivalent:*

1 *$I$ is a pure ideal of $A$.*
2 *$A/I$ is flat.*
3 *If $a \in I$ there exists an element $c \in I$ such $(1-c)a=0$*
4 *$I_{\mathfrak{m}} = 0$ or $I_{\mathfrak{m}} = R_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of $A$.*

**Lemma 2.3.** *The $(A \bowtie f, g(J, J\prime))$-module $f(A)+J$ (resp. $g(A)+J'$) is flat if and if $J'$ (resp. $J$) is a pure ideal of $g(A)+J'$ (resp. $f(A)+J$).*

**Proof.** Following Lemmas 2.1 and 2.2, $f(A)+J$ is flat if and only if, for each $(0, j_1{}') \in 0 \times J'$ there exists $((0, j_2{}') \in 0 \times J'$ such that

$$(0, (1-j\prime_2)j\prime_1) = ((1, 1) - (0, j\prime_2))\,(0, j\prime_1) = (0, 0)$$

This last condition is equivalent to that $J'$ is a pure ideal of $g(A)+J'$. The respective equivalence follows similarly. □

The main result of this section is formulated as follows:

**Theorem 2.1.** *Suppose that $J$ and $J'$ are pure ideals of $f(A)+J$ and $g(A)+J'$, respectively. Then,*

$$\mathrm{gldim}(A \bowtie^{f,g}(J, J\prime))$$
$$= \sup\{\mathrm{gldim}(f(A)+J), \mathrm{gldim}(g(A)+J\prime)\}$$

*and*

$$\mathrm{wgldim}(A \bowtie^{f,g}(J, J\prime))$$
$$= \sup\{\mathrm{wgldim}(f(A)+J), \mathrm{wgldim}(g(A)+J\prime)\}$$

**Proof.** Applying [4, Theorem 2] and [11, Corollary 6] to the fiber product

$$\begin{array}{ccc} A \bowtie^{f,g} (J, J') & \xrightarrow{\mu_1} & f(A) + J \\ \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \alpha} \\ g(A) + J' & \xrightarrow{\beta} & A/I \end{array}$$

we obtain

$$\mathrm{gldim}(A \bowtie^{f,g}(J, J\prime) \leqslant$$
$$\sup\{\mathrm{gldim}(f(A)+J), \mathrm{gldim}(g(A)+J\prime)\}$$

and

$$\mathrm{wgldim}(A \bowtie^{f,g}(J, J\prime) \leqslant$$
$$\sup\{\mathrm{wgldim}(f(A)+J), \mathrm{wgldim}(g(A)+J\prime)\}$$

since $f(A)+J$ and $g(A)+J'$ are flat $(A \bowtie f, g(J, J\prime))$-modules (by Lemma 2.3).

"$\geqslant$" For short, set $R := A \bowtie^{f,g}(J, J')$. Let $M$ be an $(f(A)+J)$-module. It is clear that $M$ can be also seen as an $R$-module via $\mu_1$ as follows:

$$(f(a)+j, g(a)+j\prime).m = (f(a)+j)m$$

For any $r \in J'$ and $m \in M$,

$$m \otimes (0, r) = m \otimes (0, r)(1, 1) = (0, r)m \otimes (1, 1)$$
$$= 0 \otimes (1, 1) = 0$$

Thus, $M \otimes_R (0 \times J')$=0. The kernel of the surjective $R$-homomorphism $1_M \otimes \mu_1$ coincides with $M \otimes_R (0 \times J')$, and thus it is an isomorphism of $R$-modules.

Consider the isomorphism of $R$-modules $\psi = (1_M \otimes \mu_1) \circ \theta : M \to M \otimes_R (f(A)+J)$ where $\theta$ is the natural isomorphism $M \to M \otimes_R R$. It is clear that $\psi$ is also an isomorphism of $(f(A)+J)$-modules. Indeed, for each $a \in A$, $j \in J$ and $m \in M$,

$$\begin{aligned}\psi((f(a)+j)m) &= (1_M \otimes \mu_1)((f(a)+j)m \otimes (1,1)) \\ &= (f(a)+j)m \otimes 1 \\ &= (f(a)+j, g(a)).m \otimes 1 \\ &= m \otimes (f(a)+j, g(a)).1 \\ &= m \otimes (f(a)+j) \\ &= (f(a)+j).(m \otimes 1) \\ &= (f(a)+j).\psi(m)\end{aligned}$$

Thus, we conclude that $M \cong M \otimes_R (f(A)+J)$ as $(f(A)+J)$-modules.

Now, suppose that $\mathrm{gldim}(R)=d<\infty$ (resp. $\mathrm{wgldim}(R)=d<\infty$) and let

$$0 \to \cdots P_d \to \ldots \to P_1 \to P_0 \to M \to 0$$

be an $R$-projective resolution of $M$ (resp. $R$-flat resolution of $M$). Since $f(A)+J$ is flat, by Lemma 2.3, we obtain the following $(f(A)+J)$-projective resolution (resp. $(f(A)+J)$-flat resolution) of $M \otimes_R (f(A)+J) \cong M$

$$0 \to P_d \otimes_R (f(A)+J) \to \cdots \to P_0 \otimes_R (f(A)+J) \\ \to M \otimes_R (f(A)+J) \to 0$$

Thus, $\mathrm{pd}_{f(A)+J}(M) \leqslant d$ (resp. $\mathrm{fd}_{f(A)+J}(M) \leqslant d$), and so $\mathrm{gldim}(f(A)+J) \leqslant d$ (resp. $\mathrm{wgldim}(f(A)+J) \leqslant d$). Similarly, $\mathrm{gldim}(g(A)+J') \leqslant d$ (resp. $\mathrm{wgldim}(g(A)+J') \leqslant d$). □

**Corollary 2.1.** *Let $f: A \to B$ be a ring homomorphism and $J$ a pure ideal of $B$ such $f^{-1}(J)$ is a pure ideal of $A$. Then,*

$$\mathrm{gldim}(A \bowtie^f J) = \sup\{\mathrm{gldim}(A), \mathrm{gldim}(f(A)+J)\}$$

*and*

$$\mathrm{wgldim}(A \bowtie^f J) \\ = \sup\{\mathrm{wgldim}(A), \mathrm{wgldim}(f(A)+J)\}$$

**Proof.** Seen that $J$ is also a pure ideal of $f(A)+J$, the result follows immediately from Theorem 2.1 since $A \bowtie^f J = A \bowtie^{\iota,f} (f^{-1}(J), J)$.□

**Corollary 2.2.** *Let $A$ be a ring and $I$ a pure ideal of $A$. Then,*

$$\mathrm{gldim}(A \bowtie I) = \mathrm{gldim}(A)$$

*and*

$$\mathrm{wgldim}(A \bowtie I) = \mathrm{wgldim}(A)$$

**Corollary 2.3.** *Let $f: A \to B$ be a ring homomorphism and $J$ a pure ideal of $B$. If $A$ is a Von Neumann regular ring, then*

$$\mathrm{gldim}(A \bowtie^f J) = \sup\{\mathrm{gldim}(A), \mathrm{gldim}(f(A)+J))\}$$

*and*

$$\mathrm{wgldim}(A \bowtie^f J) = \mathrm{wgldim}(f(A)+J)$$

**Proof.** Since $A/f^{-1}(J)$ is a flat $A$-module, $f^{-1}(J)$ is always a pure ideal of $A$. Then, this result is a particular case of Corollary 2.1.□

**Corollary 2.4.** *If $J$ and $J'$ are generated by idempotent element respectively, then*

$$\mathrm{gldim}(A \bowtie^{f,g} (J, J')) \\ = \sup\{\mathrm{gldim}(f(A)+J), \mathrm{gldim}(g(A)+J')\}$$

*and*

$$\mathrm{wgldim}(A \bowtie^{f,g} (J, J')) \\ = \sup\{\mathrm{wgldim}(f(A)+J), \mathrm{wgldim}(g(A)+J')\}$$

**Proof.** Follows from Theorem 2.1 since ideals generated by idempotent elements are pures. □

**Corollary 2.5.** *Let $A$ be a ring and $I$ an ideal of $A$ generated by idempotent element. Then,*

$$\mathrm{gldim}(A \bowtie I) = \mathrm{gldim}(A)$$

*and*

$$\mathrm{wgldim}(A \bowtie I) = \mathrm{wgldim}(A)$$

Recall that a ring $A$ is called $FP$-ring if every principal ideal of $A$ is flat.

**Proposition 2.1.** *Let $A$ be a ring and $I$ an ideal of $A$. If $A \bowtie I$ is a PF-ring then $A$ is a PF-ring and $I$ is a pure ideal of $A$.*

**Proof.** ($\Rightarrow$) Let $m$ be a maximal ideal of $A$.

If $I \subseteq m$, then $m \bowtie I$ is a maximal ideal of $A \bowtie I$, and we have

$$(A \bowtie I)_{m \bowtie I} \cong A_m \bowtie I_m$$

Following [10, Theorem 4.2.2], $A_m \bowtie I_m$ is an domain, an so $I_m = 0$ and $A_m$ is a domain.

If $I \nsubseteq m$, then $\overline{m} = \{(x, x+i) \mid x \in A, i \in I, x+i \in m\}$ is a maximal ideal of $A \bowtie I$, and we have

$$(A \bowtie I)_{\overline{m}} \cong A_m$$

and we have $I_m = A_m$. Thus, $A$ is a $PF$-ring and $I$ is a pure ideal of $A$.□

**Corollary 2.1.** *Let $A$ be a Noetherian ring, $I$ an ideal of $A$ and $d$ an integer. Then, the following are equivalent:*

1 $\mathrm{gldim}(A \bowtie I) = d$.
2 $\mathrm{gldim}(A) = d$ *and $I$ is generated by an idempotent element.*

**Proof.** Since $A \bowtie I = A \bowtie^{\iota,\iota}(I, I)$ and following Corollary 2.5, it suffices to show that (1) implies that $I$ is generated by an idempotent element. Note that $A \bowtie I$ is a Noetherian ring, and so it is a regular ring provided (1), and so a $PF$-ring. Hence, by Proposition 2.1, $I$ is a pure ideal of $A$. Thus, $A/I$ is a finitely presented flat module, and so projective. Hence, $I$ is generated by an idempotent element. □

**Corollary 2.2.** *Let $A$ be a ring, $I$ an ideal of $A$ and $d$ an integer. Suppose that $A \bowtie I$ is coherent (in particular if $A$ is coherent and $I$ is a finitely generated ideal), Then, the following are equivalent:*

1 $\mathrm{wgldim}(A \bowtie I) = d$.
2 $\mathrm{wgldim}(A) = d$ *and $I$ is a pure ideal of $A$.*

**Proof.** Follows from [10, Theorem 4.2.2 & Corollary 4.2.4.], Proposition 2.1 and Theorem 2.1. □

**Corollary 2.3.** *Let $A$ be a ring and $I$ an ideal of $A$. Then, the following are equivalent:*

1 $\mathrm{gldim}(A \bowtie I) = 2$.
2 $\mathrm{gldim}(A) = 2$ *and $I$ is a pure ideal of $A$.*

**Proof.** Follows from [10, Theorem 4.2.2 & Corollary 4.2.5], Proposition 2.1 and Theorem 2.1. □

**Corollary 2.4.** *Let $A$ be a ring and $I$ an ideal of $A$. Then, the following are equivalent:*

1 $\mathrm{wgldim}(A \bowtie I) = 1$.
2 $\mathrm{wgldim}(A) = 1$ *and $I$ is a pure ideal of $A$.*

**Proof.** Follows from [10, Theorem 4.2.2 & Corollary 4.2.6], Proposition 2.1 and Theorem 2.1. □

## References

[1] J. Milnor, Introduction to Algebraic K-theory, Annals of Mathematics Studies, Number 72, Princeton University Press, Princeton, NJ, 1971.
[2] A. Facchini, P. Vamos, Injective modules over pullbacks, J. London Math. Soc. 31 (2) (1985) 425–438.
[3] A.N. Wiseman, Projective modules over pullback rings, Math. Proc. Camb. Phil. Soc. 97 (1985) 399–406.
[4] E. Kirkman, J. Kuzmanovich, On the global dimension of fibre products, Pacific J. Math. 134 (1988) 121–132.
[5] S. Kabbaj, K. Louartiti, M. Tamekkante, Bi-amalgamated algebras along ideals, preprint, arXiv:1407.7074v1.
[6] M.B. Boisen, P.B. Sheldon, CPI-extension: overrings of integral domains with special prime spectrum, Can. J. Math. 29 (1977) 722–737.
[7] M. D'Anna, C.A. Finacchiaro, M. Fontana, Amalgamated algebras along an ideal, in: Comm Algebra and Applications, Walter De Gruyter, 2009, pp. 241–252.
[8] M. D'Anna, C.A. Finacchiaro, M. Fontana, Properties of chains of prime ideals in amalgamated algebras along an ideal, J. Pure Appl. Algebra 214 (2010) 1633–1641.
[9] M. D'Anna, M. Fontana, An amalgamated duplication of a ring along an ideal: the basic properties, J. Algebra Appl. 6 (3) (2007) 443–459.
[10] S. Glaz, Commutative Coherent Rings, Lecture Notes in Mathematics 1371, Springer-Verlag, Berlin, 1989.
[11] S. Scrivanti, Homological dimension of pullbacks, Math. Scand. 71 (1992) 5–15.